

# A short survey on $\delta$ -ideal CR submanifolds

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## Abstract

This paper surveys some of the known results on  $\delta$ -ideal CR submanifolds in complex space forms, the nearly Kähler 6-sphere and odd dimensional unit spheres. In addition, the relationship between  $\delta$ -ideal CR submanifolds and critical points of the  $\lambda$ -bienergy is mentioned. Some topics on variational problem for the  $\lambda$ -bienergy are also presented.

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## 1 Introduction

In submanifold theory, it is important to establish relations between extrinsic and intrinsic invariants of submanifolds. In the early 1990s, the notion of  $\delta$ -invariants was introduced by Chen (see [11], [13] and [14]). These invariants are obtained by subtracting a certain amount of sectional curvatures from the scalar curvature. Furthermore, he established pointwise optimal inequalities involving  $\delta$ -invariants and the squared mean curvature of arbitrary submanifolds in real and complex space forms. A submanifold is said to be  $\delta$ -ideal if it satisfies an equality case of the inequalities everywhere. During the last two decades, many interesting results on  $\delta$ -ideal submanifolds have been obtained.

The main purpose of this paper is to survey some of the known results on  $\delta$ -ideal CR submanifolds in complex space forms, the nearly Kähler 6-sphere and odd dimensional unit spheres. For a given compact almost CR manifold  $M$  (with or without boundary) equipped with a compatible metric, the  $\delta$ -ideal CR immersions of  $M$  minimize the  $\lambda$ -bienergy among all isometric CR immersions of  $M$ . In view of this fact, some topics on variational problem for the  $\lambda$ -bienergy are also presented.

## 2 Preliminaries

Let  $M$  be an  $n$ -dimensional submanifold of a Riemannian manifold  $\tilde{M}$ . Let us denote by  $\nabla$  and  $\tilde{\nabla}$  the Levi-Civita connections on  $M$  and  $\tilde{M}$ , respectively. The Gauss and Weingarten formulas are respectively given by

$$\begin{aligned}\tilde{\nabla}_X Y &= \nabla_X Y + B(X, Y), \\ \tilde{\nabla}_X V &= -A_V X + D_X V\end{aligned}$$

for tangent vector fields  $X, Y$  and normal vector field  $V$ , where  $B, A$  and  $D$  are the second fundamental form, the shape operator and the normal connection.

The mean curvature vector field  $H$  is defined by  $H = (1/n)\text{trace } B$ . The function  $|H|$  is called the *mean curvature*. If it vanishes identically, then  $M$  is called a *minimal submanifold*. In particular, if  $B$  vanishes identically, then  $M$  is called a *totally geodesic submanifold*.

**Definition 2.1** ([6]). Let  $M$  be a Riemannian submanifold of an almost Hermitian manifold  $\tilde{M}$  and let  $J$  be the complex structure of  $\tilde{M}$ . A submanifold  $M$  is called a *CR submanifold* if there exist differentiable distributions  $\mathcal{H}$  and  $\mathcal{H}^\perp$  such that

$$TM = \mathcal{H} \oplus \mathcal{H}^\perp, \quad J\mathcal{H} = \mathcal{H}, \quad J\mathcal{H}^\perp \subset T^\perp M,$$

where  $T^\perp M$  denotes the normal bundle of  $M$ . A CR submanifold is called a *Kähler submanifold* (resp. *totally real submanifold*) if  $\text{rank } \mathcal{H}^\perp = 0$  (resp.  $\text{rank } \mathcal{H} = 0$ ). A totally real submanifold is called a *Lagrangeian submanifold* if  $J(TM) = T^\perp M$ . A CR submanifold is said to be *proper* if  $\text{rank } \mathcal{H} \neq 0$  and  $\text{rank } \mathcal{H}^\perp \neq 0$ .

### 3 $\delta$ -invariants

Let  $M$  be an  $n$ -dimensional Riemannian manifold. Denote by  $K(\pi)$  the sectional curvature of  $M$  associated with a plane section  $\pi \subset T_p M$ ,  $p \in M$ . For any orthonormal basis  $\{e_1, \dots, e_n\}$  of the tangent space  $T_p M$ , the scalar curvature  $\tau$  at  $p$  is defined by

$$\tau(p) = \sum_{i < j} K(e_i \wedge e_j).$$

Let  $L$  be a subset of  $T_p M$  of dimension  $r \geq 2$  and  $\{e_1, \dots, e_r\}$  an orthonormal basis of  $L$ . We define the scalar curvature  $\tau(L)$  of the  $r$ -plane section  $L$  by

$$\tau(L) = \sum_{\alpha < \beta} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha, \beta \leq r.$$

For an integer  $k \geq 0$ , denote by  $\mathcal{S}(n, k)$  the finite set which consists of unordered  $k$ -tuples  $(n_1, \dots, n_k)$  of integers satisfying  $2 \leq n_1, \dots, n_k < n$  and  $n_1 + \dots + n_k \leq n$ . We denote by  $\mathcal{S}(n)$  the set of  $k$ -tuples with  $k \geq 0$  for a fixed  $n$ .

For each  $k$ -tuple  $(n_1, \dots, n_k) \in \mathcal{S}(n)$ , the notion of  $\delta$ -invariant  $\delta(n_1, \dots, n_k)$  was introduced by Chen [13] as follows:

$$\delta(n_1, \dots, n_k)(p) = \tau(p) - \inf\{\tau(L_1) + \dots + \tau(L_k)\},$$

where  $L_1, \dots, L_k$  run over all  $k$  mutually orthogonal subspaces of  $T_p M$  such that  $\dim L_j = n_j$ ,  $j = 1, \dots, k$ .

Let  $\overline{Ric}$  denote the maximum Ricci curvature function on  $M$  defined by

$$\overline{Ric}(p) = \max\{S(X, X) | X \in U_p M\},$$

where  $S$  is the Ricci tensor and  $U_p M$  is the unit tangent vector space of  $M$  at  $p$ . Then, we have  $\delta(n-1)(p) = \overline{Ric}(p)$ .

Let  $M$  be a Kähler manifold with real dimension  $2n$ . For each  $k$ -tuple  $(2n_1, \dots, 2n_k) \in \mathcal{S}(2n)$ , Chen [13] also introduced the notion of *complex  $\delta$ -invariant*  $\delta^c(2n_1, \dots, 2n_k)$ , which is defined by

$$\delta^c(2n_1, \dots, 2n_k)(p) = \tau(p) - \inf\{\tau(L_1^c) + \dots + \tau(L_k^c)\},$$

where  $L_1^c, \dots, L_k^c$  run over all  $k$  mutually orthogonal complex subspaces of  $T_p M$  such that  $\dim L_j = 2n_j$ ,  $j = 1, \dots, k$ .

For simplicity, we denote  $\delta(\lambda, \dots, \lambda)$  and  $\delta^c(\lambda, \dots, \lambda)$  by  $\delta_k(\lambda)$  and  $\delta_k^c(\lambda)$ , respectively, where  $\lambda$  appears  $k$  times.

#### 4 Inequalities involving $\delta$ -invariants and ideal submanifolds

For each  $(n_1, \dots, n_k) \in \mathcal{S}(n)$ , let  $c(n_1, \dots, n_k)$  and  $b(n_1, \dots, n_k)$  be the constants given by

$$c(n_1, \dots, n_k) = \frac{n^2(n+k-1 - \sum_{j=1}^k n_j)}{2(n+k - \sum_{j=1}^k n_j)},$$

$$b(n_1, \dots, n_k) = \frac{1}{2} \left( n(n-1) - \sum_{j=1}^k n_j(n_j-1) \right).$$

Chen obtained the following inequality for an arbitrary submanifold in a real space form.

**Theorem 4.1** ([14]). *Given an  $n$ -dimensional submanifold  $M$  in an  $m$ -dimensional real space form  $R^m(\epsilon)$  of constant sectional curvature  $\epsilon$ , we have*

$$\delta(n_1, \dots, n_k) \leq c(n_1, \dots, n_k)|H|^2 + b(n_1, \dots, n_k)\epsilon. \quad (4.1)$$

*Equality sign of (4.1) holds at a point  $p \in M$  for some  $(n_1, \dots, n_k) \in \mathcal{S}(n)$  if and only if there exists an orthonormal basis  $\{e_1, \dots, e_{2m}\}$  at  $p$  such that  $e_1, \dots, e_n$  are tangent to  $M$  and the shape operators of  $M$  in  $R^m(\epsilon)$  at  $p$  take following forms:*

$$A_{e_r} = \begin{pmatrix} A_1^r & \dots & 0 & & \\ \vdots & \ddots & \vdots & & 0 \\ 0 & \dots & A_k^r & & \\ & 0 & & \mu_r I & \end{pmatrix}, \quad (4.2)$$

$$r = n+1, \dots, 2m,$$

where each  $A_j^r$  is a symmetric  $n_j \times n_j$  submatrix such that

$$\text{trace}(A_1^r) = \dots = \text{trace}(A_k^r) = \mu_r. \quad (4.3)$$

Let  $\tilde{M}^m(4\epsilon)$  be a complex space form of complex dimension  $m$  and constant holomorphic sectional curvature  $4\epsilon$  and let  $J$  be the complex structure of  $\tilde{M}^m(4\epsilon)$ .

Let  $M$  be an  $n$ -dimensional submanifold in  $\tilde{M}^m(4\epsilon)$ . For any vector  $X$  tangent to  $M$ , we put  $JX = PX + FX$ , where  $PX$  and  $FX$  are tangential and normal components of  $JX$ , respectively. For a subspace  $L \subset T_p M$  of dimension  $r$ , we set

$$\Psi(L) = \sum_{1 \leq i < j \leq r} \langle Pu_i, u_j \rangle^2,$$

where  $\{u_1, \dots, u_r\}$  is an orthonormal basis of  $L$ .

For an arbitrary submanifold in a complex space form, we have

**Proposition 4.1** ([14]). *Let  $M$  be an  $n$ -dimensional submanifold in a complex space form  $\tilde{M}^m(4\epsilon)$ . Then, for mutually orthogonal subspaces  $L_1, \dots, L_k$  of  $T_p M$  such that  $\dim L_j = n_j$ , we have*

$$\tau - \sum_{i=1}^k \tau(L_i) \leq c(n_1, \dots, n_k) |H|^2 + b(n_1, \dots, n_k) \epsilon + \frac{3}{2} |P|^2 \epsilon - 3\epsilon \sum_{i=1}^k \Psi(L_i). \quad (4.4)$$

The equality case of inequality (4.4) holds at a point  $p \in M$  if and only if there exists an orthonormal basis  $\{e_1, \dots, e_{2m}\}$  at  $p$  such that

- (a)  $L_j = \text{Span}\{e_{n_1+\dots+n_{j-1}+1}, \dots, e_{n_1+\dots+n_j}\}$ ,  $j = 1, \dots, k$ ,
- (b) the shape operators of  $M$  in  $\tilde{M}^m(4\epsilon)$  at  $p$  satisfy (4.2) and (4.3).

Using Proposition 4.1, we obtain the following inequalities.

**Proposition 4.2** ([14]). *Let  $M$  be a Kähler submanifold with real dimension  $2n$  in a complex space form  $\tilde{M}^m(4\epsilon)$ . Then, we have*

$$\delta^c(2n_1, \dots, 2n_k) \leq 2 \left( n(n+1) - \sum_{j=1}^k n_j(n_j+1) \right) \epsilon. \quad (4.5)$$

The equality case of inequality (4.5) holds at a point  $p \in M$  if and only if there exists an orthonormal basis  $\{e_1, \dots, e_{2m}\}$  at  $p$  such that  $e_1, \dots, e_{2n}$  are tangent to  $M$  and  $e_{2l} = J e_{2l-1}$  ( $1 \leq l \leq k$ ), and moreover, the shape operators of  $M$  in  $\tilde{M}^m(4\epsilon)$  at  $p$  take the following forms:

$$A_{e_r} = \begin{pmatrix} A_1^r & \dots & 0 \\ \vdots & \ddots & \vdots & 0 \\ 0 & \dots & A_k^r & 0 \\ & & 0 & 0 \end{pmatrix},$$

$$r = 2n+1, \dots, 2m,$$

where each  $A_j^r$  is a symmetric  $(2n_j) \times (2n_j)$  submatrix satisfying  $\text{trace}(A_j^r) = 0$ .

**Proposition 4.3** ([42]). *Let  $M$  be an  $n$ -dimensional CR submanifold with  $\text{rank } \mathcal{H} = 2h$  in  $\mathbb{C}H^m(-4)$ . Then, we have*

$$\delta(n_1, \dots, n_k) \leq c(n_1, \dots, n_k) |H|^2 - b(n_1, \dots, n_k) - 3h + \frac{3}{2} \sum_{j=1}^k n_j. \quad (4.6)$$

Equality sign of (4.6) holds at a point  $p \in M$  for some  $(n_1, \dots, n_k) \in \mathcal{S}(n)$  if and only if there exists an orthonormal basis  $\{e_1, \dots, e_{2m}\}$  at  $p$  such that

- (a) each  $L_j = \text{Span}\{e_{n_1+\dots+n_{j-1}+1}, \dots, e_{n_1+\dots+n_j}\}$  satisfies  $\Psi(L_j) = n_j/2$  for  $1 \leq j \leq k$ ,
- (b) the shape operators of  $M$  in  $\mathbb{C}H^m(-4)$  at  $p$  satisfy (4.2) and (4.3).

**Proposition 4.4** ([46]). *Let  $M$  be an  $n$ -dimensional CR submanifold with  $\text{rank } \mathcal{H} = 2h$  in  $\mathbb{C}P^m(4)$ . Then, we have*

$$\delta(n_1, \dots, n_k) \leq c(n_1, \dots, n_k)|H|^2 + b(n_1, \dots, n_k) + 3h. \quad (4.7)$$

*Equality sign of (4.7) holds at a point  $p \in M$  for some  $(n_1, \dots, n_k) \in \mathcal{S}(n)$  if and only if there exists an orthonormal basis  $\{e_1, \dots, e_{2m}\}$  at  $p$  such that*

- (a) *each  $L_j = \text{Span}\{e_{n_1+\dots+n_{j-1}+1}, \dots, e_{n_1+\dots+n_j}\}$  satisfies  $\Psi(L_j) = 0$  for  $1 \leq i \leq k$ ,*
- (b) *the shape operators of  $M$  in  $\mathbb{C}P^m(4)$  at  $p$  satisfy (4.2) and (4.3).*

**Definition 4.1.** A submanifold is said to be  $\delta(n_1, \dots, n_k)$ -ideal if it satisfies the equality case of (4.1), (4.6) or (4.7) identically for a  $k$ -tuple  $(n_1, \dots, n_k) \in \mathcal{S}(n)$ . Similarly, a Kähler submanifold is said to be  $\delta^c(2n_1, \dots, 2n_k)$ -ideal if it satisfies the equality case of (4.5) identically for a  $k$ -tuple  $(2n_1, \dots, 2n_k) \in \mathcal{S}(2n)$ .

For more information on  $\delta$ -invariants and  $\delta$ -ideal submanifolds, we refer the reader to [16].

**Definition 4.2.** A submanifold is said to be *linearly full* in  $\tilde{M}^m(4\epsilon)$  if it does not lie in any totally geodesic Kähler hypersurfaces of  $\tilde{M}^m(4\epsilon)$ .

## 5 Ideal CR submanifolds in complex hyperbolic space

We first recall some basic definitions on hypersurfaces.

**Definition 5.1.** Let  $N$  be a submanifold in a Riemannian manifold  $\tilde{M}$  and  $UN^\perp$  the unit normal bundle of  $N$ . Then, for a sufficiently small  $r > 0$ , the following mapping is an immersion:

$$f_r : UN^\perp \rightarrow \tilde{M}, \quad f_r(p, V) = \exp_p(rV),$$

where  $\exp$  denotes the exponential mapping of  $\tilde{M}$ . The hypersurface  $f_r(UN^\perp)$  of  $\tilde{M}$  is called the *tubular hypersurface* over  $N$  with radius  $r$ . If  $N$  is a point  $x$  in  $\tilde{M}$ , then the tubular hypersurface over  $x$  is a geodesic hypersphere centered at  $x$ .

**Definition 5.2.** For a given point  $p \in \mathbb{C}H^m(-4)$ , let  $\gamma(t)$  be a geodesic with  $\gamma(0) = p$ , which is parametrized by arch length. Denote by  $S_t(\gamma(t))$  the geodesic hypersphere centered at  $\gamma(t)$  with radius  $t$ . The limit of  $S_t(\gamma(t))$  when  $t$  tends to infinity is called a *horosphere*.

**Definition 5.3.** Let  $M$  be a real hypersurface in an almost Hermitian manifold and  $V$  be a unit normal vector. A hypersurface  $M$  is called a *Hopf hypersurface* if  $JV$  is a principal curvature vector.

A real hypersurface in an almost Hermitian manifold is a proper CR submanifold with  $\text{rank } \mathcal{H}^\perp = 1$ . The following theorem characterizes the horosphere of  $\mathbb{C}H^m(-4)$  in terms of  $\delta_k(2)$ .

**Theorem 5.1** ([14]). *Let  $M$  be a  $\delta_k(2)$ -ideal real hypersurface of  $\mathbb{C}H^m(-4)$ . Then  $k = m - 1$  and  $M$  is an open portion of the horosphere in  $\mathbb{C}H^m(-4)$ .*

**Remark 5.1.** The third case of (9.5) in [14] does not occur, because  $L_1 \dots, L_k$  are complex planes. Therefore, case (1) of Theorem 9.1 in [14] shall be removed from the list of  $\delta_k(2)$ -ideal real hypersurfaces in  $\mathbb{C}H^m(-4)$ .

For  $\delta(2m-2)$ -ideal real hypersurfaces in  $\mathbb{C}H^m(-4)$ , Chen proved the following.

**Theorem 5.2** ([15]). *Let  $M$  be a real hypersurface of  $\mathbb{C}H^m(-4)$ . Then  $M$  is  $\delta(2m-2)$ -ideal if and only if  $M$  is a Hopf hypersurface with constant mean curvature given by  $2\alpha/(2m-1)$ , where  $A_V JV = \alpha JV$  for a unit normal vector  $V$ . If  $M$  has constant principal curvatures, then  $M$  is an open portion of one of the following real hypersurfaces:*

- (1) *the horosphere of  $\mathbb{C}H^2(-4)$ ;*
- (2) *the tubular hypersurface over totally geodesic  $\mathbb{C}H^{m-1}(-4)$  in  $\mathbb{C}H^m(-4)$  with radius  $r = \tanh^{-1}(1/\sqrt{2m-3})$ , where  $m \geq 3$ .*

It was proved in [15] that if  $m = 2$  in Theorem 5.2, then the assumption of the constancy of principal curvatures is satisfied. That is to say, we have

**Corollary 5.1** ([15]). *Let  $M$  be a  $\delta(2)$ -ideal real hypersurface of  $\mathbb{C}H^2(-4)$ . Then  $M$  is an open portion of the horosphere.*

Let  $\mathbb{C}_1^{m+1}$  be the complex number  $(m+1)$ -space endowed with the complex coordinates  $(z_0, \dots, z_m)$ , the pseudo-Euclidean metric given by  $\tilde{g} = -dz_0 d\bar{w}_0 + \sum_{i=1}^m dz_i d\bar{w}_i$  and the standard complex structure. For  $\epsilon < 0$ , we put  $H_1^{2m+1}(\epsilon) = \{z \in \mathbb{C}_1^{m+1} \mid \langle z, z \rangle = 1/\epsilon\}$ , where  $\langle, \rangle$  denotes the inner product on  $\mathbb{C}_1^{m+1}$  induced from  $\tilde{g}$ . For a given  $z \in H_1^{2m+1}(\epsilon)$ , we put  $[z] = \{\lambda z \mid \lambda \in \mathbb{C}, \lambda \bar{\lambda} = 1\}$ . The Hopf fibration is given by

$$\varpi_{\{m, \epsilon\}} : H_1^{2m+1}(\epsilon) \rightarrow \mathbb{C}H^m(4\epsilon) : z \mapsto [z].$$

For  $\delta_k(2)$ -ideal proper CR submanifolds in  $\mathbb{C}H^m(-4)$  whose codimensions are greater than one, we have the following representation formula.

**Theorem 5.3** ([40]). *Let  $M$  be a linearly full  $(2n+1)$ -dimensional  $\delta_k(2)$ -ideal CR submanifold in  $\mathbb{C}H^m(-4)$  such that  $\text{rank } \mathcal{H}^\perp = 1$ ,  $k \geq 1$  and  $m > n+1$ . Then, up to holomorphic isometries of  $\mathbb{C}H^m(-4)$ , the immersion of  $M$  into  $\mathbb{C}H^m(-4)$  is given by the composition  $\varpi_{\{m, -1\}} \circ z$ , where*

$$z = \left( -1 - \frac{1}{2}|\Psi|^2 + iu, -\frac{1}{2}|\Psi|^2 + iu, \Psi \right) e^{it}, \quad (5.1)$$

and  $\Psi$  is a  $2n$ -dimensional  $\delta_n^c(2)$ -ideal Kähler submanifold in  $\mathbb{C}^{m-1}$ .

Up to holomorphic isometries of  $\mathbb{C}H^m(-4)$ , the horosphere in  $\mathbb{C}H^m(-4)$  is a real hypersurface defined by  $\{[z] : z \in H_1^{2m+1}(-1), |z_0 - z_1| = 1\}$  (see, for example, [49]). Hence, Theorem 5.3 can be considered as an extension of Theorem 5.1.

As an immediate corollary of Theorem 5.3, we obtain

**Corollary 5.2** ([19]). *Let  $M$  be a linearly full 3-dimensional  $\delta(2)$ -ideal CR submanifold in  $\mathbb{C}H^m(-4)$  such that  $\text{rank } \mathcal{H}^\perp = 1$  and  $m > 2$ . Then, up to holomorphic isometries of  $\mathbb{C}H^m(-4)$ , the immersion of  $M$  into  $\mathbb{C}H^m(-4)$  is given by the composition  $\varpi_{\{m, -1\}} \circ z$ , where  $z$  is given by (5.1) and  $\Psi(w)$  is a holomorphic curve in  $\mathbb{C}^{m-1}$  with  $\Psi'(w) \neq 0$ .*

For general  $\delta(n_1, \dots, n_k)$ -ideal proper CR submanifolds in  $\mathbb{C}H^m(-4)$  whose codimensions are greater than one, the following classification result has been obtained.

**Theorem 5.4** ([42], [46]). *Let  $M$  be a linearly full  $(2n+1)$ -dimensional  $\delta(n_1, \dots, n_k)$ -ideal CR submanifold in  $\mathbb{C}H^m(-4)$  such that  $\text{rank } \mathcal{H}^\perp = 1$ ,  $k \geq 1$  and  $m > n + 1$ . Then, we have  $JH \in \mathcal{H}^\perp$ ,  $A_V JV = (2n/\sqrt{k(2n-k)})JV$  for  $V = H/|H|$ ,  $DH = 0$ , and moreover, the mean curvature is given by*

$$\frac{2n(k+1)}{(2n+1)\sqrt{k(2n-k)}}.$$

*If all principal curvatures of  $M$  with respect to  $H/|H|$  are constant, then one of the following two cases occurs:*

- (1)  *$M$  is locally congruent with the immersion described in Theorem 5.3.*
- (2)  *$n/k \in \mathbb{Z} - \{1\}$ ,  $n_1 = \dots = n_k = 2n/k$ , and  $M$  is locally congruent with the immersion*

$$\varpi_{\{m,-1\}} \left( \varpi_{\{m-1, \frac{2k-2n}{2n-k}\}}^{-1}(\Psi), \sqrt{\frac{k}{2n-2k}} e^{it} \right),$$

*where  $\Psi$  is a  $2n$ -dimensional  $\delta_k^c(2n/k)$ -ideal Kähler submanifold in  $\mathbb{C}H^{m-1}(\frac{8k-8n}{2n-k})$ .*

If  $n > 1$ ,  $k = 1$  and  $n_1 = 2n$  in Theorem 5.4, then we have

**Corollary 5.3** ([41], [46]). *Let  $M$  be a linearly full  $(2n+1)$ -dimensional CR submanifold in  $\mathbb{C}H^m(-4)$  such that  $\text{rank } \mathcal{H}^\perp = 1$ ,  $n > 1$  and  $m > n + 1$ . Then  $M$  is  $\delta(2n)$ -ideal if and only if  $JH \in \mathcal{H}^\perp$ ,  $DH = 0$ ,  $A_V JV = (2n/\sqrt{(2n-1)})JV$  for  $V = H/|H|$ , and moreover, the mean curvature is given by*

$$\frac{4n}{(2n+1)\sqrt{2n-1}}.$$

*If all principal curvatures of  $M$  with respect to  $H/|H|$  are constant, then, up to holomorphic isometries of  $\mathbb{C}H^m(-4)$ , the immersion of  $M$  into  $\mathbb{C}H^m(-4)$  is given by*

$$\varpi_{\{m,-1\}} \left( \varpi_{\{m-1, \frac{2-2n}{2n-1}\}}^{-1}(\Psi), \sqrt{\frac{1}{2n-2}} e^{it} \right),$$

*where  $\Psi$  is a  $2n$ -dimensional Kähler submanifold in  $\mathbb{C}H^{m-1}(\frac{8-8n}{2n-1})$ .*

A hypersurface given by (2) in Theorem 5.2 can be rewritten as follows (see, for example, [37, Example 6.1]):

$$\varpi_{\{m,-1\}} \left( H_1^{2m-1} \left( \frac{4-2m}{2m-3} \right) \times S^1 \left( \frac{1}{\sqrt{2m-4}} \right) \right),$$

where  $S^1(r) = \{z \in \mathbb{C} | z\bar{z} = r^2\}$ . Thus, Corollary 5.3 can be regarded as an extension of Theorem 5.2.

Let  $N$  be a Kähler hypersurface with real dimension  $2n$  in a complex space form. Let  $V$  and  $JV$  be normal vector fields of  $N$ . Since  $A_{JV} = JA_V$  and  $JA_V = -A_V J$  holds (cf. [34, p.175]), there exists an orthonormal basis  $\{e_1, Je_1, \dots, e_n, Je_n\}$



of  $T_p N$  with respect to which the shape operators  $A_V$  and  $A_{JV}$  take the following forms:

$$A_V = \begin{pmatrix} \lambda_1 & & & 0 \\ & -\lambda_1 & & \\ & & \ddots & \\ & & & \lambda_n \\ 0 & & & & -\lambda_n \end{pmatrix}, \quad A_{JV} = \begin{pmatrix} 0 & \lambda_1 & & 0 \\ \lambda_1 & 0 & & \\ & & \ddots & \\ & & & 0 & \lambda_n \\ 0 & & & \lambda_n & 0 \end{pmatrix}.$$

Hence, it follows from Proposition 4.2 that every Kähler hypersurface with real dimension  $2n$  in a complex space form is  $\delta_k^c(2n/k)$ -ideal for any natural number  $k$  such that  $n/k \in \mathbb{Z}$ . Accordingly, applying Theorem 5.3 yields the following.

**Corollary 5.4** ([40]). *Let  $M$  be a linearly full  $(2n+1)$ -dimensional  $\delta_k(2)$ -ideal CR submanifold in  $\mathbb{C}H^{n+2}(-4)$  such that  $\text{rank } \mathcal{H}^\perp = 1$  and  $k \geq 1$ . Then, up to holomorphic isometries of  $\mathbb{C}H^{n+2}(-4)$ , the immersion of  $M$  into  $\mathbb{C}H^{n+2}(-4)$  is given by the composition  $\varpi_{\{n+2,-1\}} \circ z$ , where  $z$  is given by (5.1), and  $\Psi$  is a Kähler hypersurface in  $\mathbb{C}^{n+1}$ .*

Similarly, we obtain the following corollary of Theorem 5.4.

**Corollary 5.5.** *Let  $M$  be a linearly full  $(2n+1)$ -dimensional  $\delta(n_1, \dots, n_k)$ -ideal CR submanifold in  $\mathbb{C}H^{n+2}(-4)$  such that  $\text{rank } \mathcal{H}^\perp = 1$  and  $k \geq 1$ . If all principal curvatures of  $M$  with respect to  $H/|H|$  are constant, then one of the following two cases occurs:*

- (1)  $M$  is locally congruent with the immersion described in Corollary 5.4.
- (2)  $n/k \in \mathbb{Z} - \{1\}$ ,  $n_1 = \dots = n_k = 2n/k$ , and  $M$  is locally congruent with the immersion

$$\varpi_{\{n+2,-1\}} \left( \varpi_{\{n+1, \frac{2k-2n}{2n-k}\}}^{-1}(\Psi), \sqrt{\frac{k}{2n-2k}} e^{it} \right),$$

where  $\Psi$  is a Kähler hypersurface in  $\mathbb{C}H^{n+1}(\frac{8k-8n}{2n-k})$ .

It is natural to ask the following problem.

**Problem 5.1.** *Find  $\delta(n_1, \dots, n_k)$ -ideal CR submanifolds with  $\text{rank } \mathcal{H}^\perp = 1$  in  $\mathbb{C}H^m(-4)$  such that the principal curvatures with respect to  $H/|H|$  are not all constant.*

Generally,  $\delta(n_1, \dots, n_k)$ -ideal proper CR submanifolds in  $\mathbb{C}H^m(-4)$  have the following properties.

**Theorem 5.5** ([42]). *Let  $M$  be a linearly full  $(2n+q)$ -dimensional  $\delta(n_1, \dots, n_k)$ -ideal CR submanifold in  $\mathbb{C}H^m(-4)$  such that  $\text{rank } \mathcal{H}^\perp = q$ . If  $q > 1$ , then  $M$  is minimal. If  $q = 1$  and  $m > n+1$ , then  $M$  is non-minimal and satisfies  $DH = 0$ .*

A differentiable manifold  $M$  is called an *almost contact manifold* if it admits a unit vector field  $\xi$ , a one-form  $\eta$  and a  $(1,1)$ -tensor field  $\phi$  satisfying

$$\eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi.$$

Every almost contact manifold admits a Riemannian metric  $g$  satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$



The quadruplet  $(\phi, \xi, \eta, g)$  is called an *almost contact metric structure*.

An almost contact metric structure is called a *contact metric structure* if it satisfies

$$d\eta(X, Y) = \frac{1}{2} \left( X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]) \right) = g(X, \phi Y).$$

A contact metric structure is said to be *Sasakian* if the tensor field  $S$  defined by

$$S(X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + 2d\eta(X, Y)\xi$$

vanishes identically. A manifold equipped with a Sasakian structure is called a *Sasakian manifold*. We refer the reader to [8] for more information on Sasakian manifolds.

Let  $M$  be a CR submanifold with rank  $\mathcal{H}^\perp = 1$  in a complex space form. We define a one-form  $\eta$  by  $\eta(X) = g(U, X)$ , where  $U$  is a unit tangent vector field lying in  $\mathcal{H}^\perp$ , and  $g$  is an induced metric on  $M$ . We put  $\bar{U} = (1/\sqrt{r})U$ ,  $\bar{\eta} = \sqrt{r}\eta$  and  $\bar{g} = rg$  for a positive constant  $r$ . Then, the quadruplet  $(P, \bar{U}, \bar{\eta}, \bar{g})$  defines an almost contact structure on  $M$  (cf. [21, p.96]).

For the almost contact structure  $(P, \bar{U}, \bar{\eta}, \bar{g})$  on a CR submanifold described in Theorem 5.4, we have the following.

**Proposition 5.1** ([46]). *An almost contact structure  $(P, \bar{U}, \bar{\eta}, \bar{g})$  with  $r = \sqrt{\frac{k}{2n-k}}$  on a CR submanifold in Theorem 5.4. becomes a Sasakian structure. In particular, in the case of (1), the structure is Sasakian with respect to the induced metric.*

## 6 Ideal CR submanifolds in complex projective space

All  $\delta_k(2)$ -ideal Hopf hypersurfaces of  $\mathbb{C}P^m(4)$  have been determined as follows:

**Theorem 6.1** ([14]). *Let  $M$  be a  $\delta_k(2)$ -ideal Hopf hypersurface of  $\mathbb{C}P^m(4)$ . Then, one of the following three cases occurs:*

- (1)  $k = 1$  and  $M$  is an open portion of a geodesic sphere with radius  $\pi/4$ ;
- (2)  $m$  is odd,  $k = m - 1$ , and  $M$  is an open portion of a tubular hypersurface with radius  $r \in (0, \pi/2)$  over a totally geodesic  $\mathbb{C}P^{(m-2)/2}(4)$ ;
- (3)  $m = 2$ ,  $k = 1$ , and  $M$  is an open portion of a tubular hypersurface over the complex quadric curve  $Q_1 := \{[z_0, z_1, z_2] \in \mathbb{C}P^2 : z_0^2 + z_1^2 + z_2^2 = 0\}$ , with radius  $r = \tan^{-1}((1 + \sqrt{5} - \sqrt{2 + 2\sqrt{5}})/2) = 0.33311971 \dots$ . Here,  $[z_0, z_1, z_2]$  is a homogeneous coordinate of  $\mathbb{C}P^2$ .

A real hypersurface of  $\mathbb{C}P^m(4)$  is called a *ruled real hypersurface* if  $\mathcal{H}$  is integrable and each leaf of its maximal integral manifolds is locally congruent to  $\mathbb{C}P^{m-1}(4)$ . For a unit normal vector  $V$  of a ruled real hypersurface  $M$ , the shape operator  $A_V$  satisfies

$$A_V JV = \mu JV + \nu U \quad (\nu \neq 0), \quad AU = \nu JV, \quad AX = 0 \quad (6.1)$$

for all  $X$  orthogonal to both  $JV$  and  $U$ , where  $U$  is a unit vector orthogonal to  $JV$ , and  $\mu$  and  $\nu$  are smooth functions on  $M$ . Thus, all ruled real hypersurfaces of  $\mathbb{C}P^m(4)$  are non-Hopf (see [33]).

Using Proposition 4.4 and (6.1), we find that every minimal ruled real hypersurface in  $\mathbb{C}P^m(4)$  is  $\delta_k(2)$ -ideal for  $1 \leq k \leq m - 1$ . Such a hypersurface can be represented as follows:

**Theorem 6.2** ([1]). *A minimal ruled hypersurface of  $\mathbb{C}P^m(4)$  is congruent to  $\varpi \circ z$ , where  $\varpi : S^{2m+1}(1) \rightarrow \mathbb{C}P^m(4)$  is the Hopf fibration and*

$$z(s, t, \theta, w) = e^{\sqrt{-1}\theta} \left( \cos s \cos t, \cos s \sin t, (\sin s)w \right)$$

for  $w \in \mathbb{C}^{m-1}$ ,  $|w|^2 = 1$ ,  $-\pi/2 < s < \pi/2$ ,  $0 \leq t, \theta < 2\pi$ .

It seems interesting to consider the following problem.

**Problem 6.1.** *Classify  $\delta(n_1, \dots, n_k)$ -ideal non-Hopf real hypersurfaces in  $\mathbb{C}P^m(4)$ .*

Let  $M$  be an  $n$ -dimensional  $\delta(n_1, \dots, n_k)$ -ideal CR submanifold in  $\mathbb{C}P^m(4)$ . Let  $L_j$  be subspaces of  $T_p M$  defined in (a) of Proposition 4.4. Define the subspace  $L_{k+1}$  by  $L_{k+1} = \text{Span}\{e_{n_1+\dots+n_k+1}, \dots, e_n\}$ . It is clear that  $T_p M = L_1 \oplus \dots \oplus L_{k+1}$ . We denote by  $\mathcal{L}_i$  the distribution which is generated by  $L_i$ . Then, we have the following codimension reduction theorem.

**Theorem 6.3** ([46]). *Let  $M$  be an  $n$ -dimensional  $\delta(n_1, \dots, n_k)$ -ideal CR submanifold with  $\text{rank } \mathcal{H}^\perp = 1$  in  $\mathbb{C}P^m(4)$ . If  $\mathcal{H}^\perp \subset \mathcal{L}_i$  for some  $i \in \{1, \dots, k+1\}$ , then  $M$  is contained in a totally geodesic Kähler submanifold  $\mathbb{C}P^{\frac{n+1}{2}}(4)$  in  $\mathbb{C}P^m(4)$ .*

It was proved in [46] that if  $\dim M = 3$ , then the assumption on  $\mathcal{H}^\perp$  in Theorem 6.3 holds. That is to say, we have the following.

**Corollary 6.1** ([46]). *Let  $M$  be a 3-dimensional  $\delta(2)$ -ideal proper CR submanifold in  $\mathbb{C}P^m(4)$ . Then,  $M$  is contained in  $\mathbb{C}P^2(4)$ .*

The following problem arises naturally.

**Problem 6.2.** *Find  $\delta(n_1, \dots, n_k)$ -ideal CR submanifolds with  $\text{rank } \mathcal{H}^\perp = 1$  in  $\mathbb{C}P^m(4)$  such that the codimensions are greater than one.*

## 7 Ideal CR submanifolds in the nearly Kähler 6-sphere

Let  $\mathcal{O}$  be the Cayley algebra, and denote by  $\text{Im } \mathcal{O}$  the purely imaginary part of  $\mathcal{O}$ . We identify  $\text{Im } \mathcal{O}$  with  $\mathbb{R}^7$  and define the exterior product  $u \times v$  on it by

$$u \times v = \frac{1}{2}(uv - vu).$$

The canonical inner product on  $\mathbb{R}^7$  is given by  $\langle u, v \rangle = -(uv + vu)/2$ .

We define the tensor field  $J$  of type  $(1, 1)$  on  $S^6(1) = \{p \in \text{Im } \mathcal{O} \mid \langle p, p \rangle = 1\}$  by

$$JX = p \times X$$

for any  $p \in S^6(1)$ ,  $X \in T_p S^6(1)$ . Let  $g$  be the standard metric on  $S^6(1)$ . Then  $(S^6(1), J, g)$  is a nearly Kähler manifold, i.e., an almost Hermitian manifold satisfying  $(\nabla_X J)X = 0$  for any  $X \in TS^6(1)$ , where  $\nabla$  is the Levi-Civita connection with respect to  $g$  (cf. [34, pp.139-140]).

For 3-dimensional  $\delta(2)$ -ideal proper CR submanifolds in the nearly Kähler  $S^6(1)$ , we have the following result.

**Theorem 7.1** ([22], [23]). *Let  $M$  be a 3-dimensional  $\delta(2)$ -ideal proper CR submanifold in the nearly Kähler  $S^6(1)$ . Then,  $M$  is minimal and locally congruent with the following immersion:*

$$f(t, u, v) = (\cos t \cos u \cos v, \sin t, \cos t \sin u \cos v, \cos t \cos u \sin v, 0, -\cos t \sin u \sin v, 0). \quad (7.1)$$

**Remark 7.1.** A CR submanifold (7.1) can be rewritten as

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_6^2 = 1, \quad x_5 = x_7 = 0, \quad x_3x_4 + x_1x_6 = 0,$$

which implies that it lies in  $S^4(1)$ .

The following theorem determines 4-dimensional  $\delta(2)$ -ideal proper CR submanifolds in the nearly Kähler  $S^6(1)$ .

**Theorem 7.2** ([2], [3]). *Let  $M$  be a 4-dimensional  $\delta(2)$ -ideal proper CR submanifold in the nearly Kähler  $S^6(1)$ . Then,  $M$  is minimal and locally congruent with the following immersion:*

$$f(t, u, v, w) = (\cos w \cos t \cos u \cos v, \sin w \sin t \cos u \cos v, \sin 2w \sin v \cos u + \cos 2w \sin u, 0, \sin w \cos t \cos u \cos v, \cos w \sin t \cos u \cos v, \cos 2w \sin v \cos u - \sin 2w \sin u). \quad (7.2)$$

**Remark 7.2.** A CR submanifold given by (7.2) lies in  $S^5(1)$ .

**Definition 7.1.** A 2-dimensional submanifold  $N$  of the nearly Kähler  $S^6(1)$  is called an *almost complex curve* if  $J(T_p N) = T_p N$  for any  $p \in N$ .

Chen has classified  $\delta_k(n_1, \dots, n_k)$ -ideal Hopf hypersurfaces of the nearly Kähler  $S^6(1)$  as follows:

**Theorem 7.3** ([16, p.415]). *A Hopf hypersurface of the nearly Kähler  $S^6(1)$  is  $\delta(n_1, \dots, n_k)$ -ideal if and only if it is either*

- (1) *a totally geodesic hypersurface, or*
- (2) *an open part of a tubular hypersurface with radius  $\pi/2$  over a non-totally geodesic almost complex curve of  $S^6(1)$ .*

**Remark 7.3.** A tubular hypersurface described in (2) of Theorem 7.3 is a minimal  $\delta(\lambda)$ -ideal hypersurface for  $\lambda \in \{2, 3, 4\}$ .

It is natural to consider the following problem.

**Problem 7.1.** *Classify 4-dimensional  $\delta(2, 2)$ -ideal and  $\delta(3)$ -ideal CR submanifolds in the nearly Kähler  $S^6(1)$ .*

## 8 Ideal contact CR submanifolds in odd dimensional unit spheres

For any point  $x \in S^{2n+1}(1) \subset \mathbb{C}^{n+1}$ , we set  $\xi = Jx$ , where  $J$  denotes the canonical complex structure of  $\mathbb{C}^{n+1}$ . Let  $g$  be the standard metric on  $S^{2n+1}(1)$  and  $\eta$  be the one-form given by  $\eta(X) = g(X, \xi)$ . We consider the orthogonal projection  $P : T_x \mathbb{C}^{n+1} \rightarrow T_x S^{2n+1}(1)$ . We define a  $(1, 1)$ -tensor field  $\phi$  on  $S^{2n+1}(1)$  by  $\phi = P \circ J$ . Then, the quadruplet  $(\phi, \xi, \eta, g)$  is a Sasakian structure (see, for example, [7]).

**Definition 8.1** ([48]). Let  $M$  be a Riemannian submanifold tangent to  $\xi$  of a Sasakian manifold. A submanifold  $M$  is called a *contact CR submanifold* if there exist differentiable distributions  $\mathcal{H}$  and  $\mathcal{H}^\perp$  such that

$$TM = \mathbb{R}\xi \oplus \mathcal{H} \oplus \mathcal{H}^\perp, \quad \phi\mathcal{H} = \mathcal{H}, \quad \phi\mathcal{H}^\perp \subset T^\perp M.$$

A contact CR submanifold is said to be *proper* if  $\text{rank } \mathcal{H} \neq 0$  and  $\text{rank } \mathcal{H}^\perp \neq 0$ .

Non-minimal  $\delta(2)$ -ideal submanifolds in a sphere have been completely described in [20]. For minimal  $\delta(2)$ -ideal proper contact CR submanifolds in  $S^{2m+1}(1)$ , we have the following codimension reduction theorem.

**Theorem 8.1** ([38]). *Let  $M^n$  be a minimal  $\delta(2)$ -ideal proper contact CR submanifold in  $S^{2m+1}(1)$ . Then  $n$  is even and there exists a totally geodesic Sasakian  $S^{2n+1}(1)$  in  $S^{2m+1}(1)$  containing  $M^n$  as a hypersurface.*

Therefore, it is sufficient to investigate the case of hypersurfaces. Let  $N$  be a minimal surface in  $S^n(1)$  and let  $UN^\perp$  be its unit normal bundle. Then, a map

$$F : UN^\perp \rightarrow S^n(1) : V_p \mapsto V_p$$

is a minimal  $\delta(2)$ -ideal codimension one immersion (see [13, Example 9.8]).

Munteanu and Vrancken proved the following.

**Theorem 8.2** ([38]). *Let  $M^{2n}$  be a minimal  $\delta(2)$ -ideal proper contact CR hypersurface in  $S^{2n+1}(1)$ . Then  $M^{2n}$  can be locally considered as the unit normal bundle of the Clifford torus  $S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2}) \subset S^3(1) \subset S^{2n+1}(1)$ .*

## 9 Related topics

This section gives an account of the relationship between  $\delta(n_1, \dots, n_k)$ -ideal immersions and critical points of the  $\lambda$ -bienergy functional  $E_{2,\lambda}$ . Some topics about variational problems for  $E_{2,\lambda}$  are also presented.

### 9.1 $\lambda$ -bienergy functional

Let  $f : M \rightarrow N$  be a smooth map of an  $n$ -dimensional Riemannian manifold into another Riemannian manifold. The *tension field*  $\tau(f)$  of  $f$  is a section of the induced vector bundle  $f^*TN$  defined by

$$\tau(f) = \sum_{i=1}^n \{ \nabla_{e_i}^f df(e_i) - df(\nabla_{e_i} e_i) \}$$

for a local orthonormal frame  $\{e_i\}$  on  $M$ , where  $\nabla^f$  and  $\nabla$  denote the induced connection and the Levi-Civita connection of  $M$ , respectively. If  $f$  is an isometric immersion, then we have

$$\tau(f) = nH. \tag{9.1}$$

A smooth map  $f$  is called a *harmonic map* if it is a critical point of the energy functional

$$E(f) = \int_{\Omega} |df|^2 dv_g$$

over every compact domain  $\Omega$  of  $M$ , where  $dv_g$  is the volume form of  $M$ . A smooth map  $f$  is harmonic if and only if  $\tau(f)$  vanishes identically on  $M$ .

**Definition 9.1.** For each smooth map  $f$  of a compact domain  $\Omega$  of  $M$  into  $N$ , the  $\lambda$ -bienergy functional is defined by

$$E_{2,\lambda}(f) = \int_{\Omega} |\tau(f)|^2 dv_g + \lambda E(f).$$

For simplicity, we denote  $E_{2,0}(f)$  by  $E_2(f)$ , which is called the bienergy functional.

Eliasson [24] proved that  $E_{2,\lambda}$  satisfies Condition (C) of Palais-Smale if the dimension of the domain is 2 or 3 and the target is non-positively curved. In general,  $E_{2,\lambda}$  does not satisfy Condition (C) (see [35]).

## 9.2 Ideal CR immersions as critical points of $\lambda$ -bienergy functional

Let  $(M, HM, J_H, g)$  be a compact Riemannian almost CR manifold (with or without boundary) whose CR dimension is  $h$ , i.e., a compact smooth manifold equipped with a subbundle  $HM$  of  $TM$  of rank  $2h$  together with a bundle isomorphism  $J_H : HM \rightarrow HM$  such that  $(J_H)^2 = -I$ , and a Riemannian metric  $g$  such that  $g(X, Y) = g(J_H X, J_H Y)$  for all  $X, Y \in HM$ .

An immersion  $f$  of  $(M, HM, J_H, g)$  into  $\tilde{M}^m(4\epsilon)$  is called a *CR immersion* if  $J(df(X)) = df(J_H(X))$  for any  $X \in HM$ . If  $f$  is an isometric immersion, then  $f(M)$  is a CR submanifold of  $\tilde{M}^m(4\epsilon)$ . We denote by  $\mathcal{ICR}(M, \tilde{M}^m(4\epsilon))$  the family of isometric CR immersions of  $M$  into  $\tilde{M}^m(4\epsilon)$ . By Proposition 4.3 and 4.4, we see that a  $\delta(n_1, \dots, n_k)$ -ideal CR immersion of  $M$  into  $\tilde{M}^m(4\epsilon)$  is a stable critical point of  $E_{2,\lambda}$  within the class of  $\mathcal{ICR}(M, \tilde{M}^m(4\epsilon))$ .

## 9.3 $\lambda$ -biharmonic submanifolds and their extensions

**Definition 9.2** ([25]). A smooth map  $f : M \rightarrow N$  is called a  $\lambda$ -biharmonic map if it is a critical point of the  $\lambda$ -bienergy functional with respect to all variations with compact support. If  $f$  is a  $\lambda$ -biharmonic isometric immersion, then  $M$  is called a  $\lambda$ -biharmonic submanifold in  $N$ . In the case of  $\lambda = 0$ , we simply call it a *biharmonic submanifold*.

The Euler-Lagrange equation for  $E_{2,\lambda}$  is given by (see [30] and [25, p.515])

$$\tau_{2,\lambda} := -\Delta_f(\tau(f)) + \text{trace} R^N(\tau(f), df)df - \lambda \tau(f) = 0, \quad (9.2)$$

where  $\Delta_f = -\sum_{i=1}^n (\nabla_{e_i}^f \nabla_{e_i}^f - \nabla_{\nabla_{e_i} e_i}^f)$  and  $R^N$  is the curvature tensor of  $N$ , which is defined by

$$R^N(X, Y)Z = [\nabla_X^N, \nabla_Y^N]Z - \nabla_{[X, Y]}^N Z$$

for the Levi-Civita connection  $\nabla^N$  of  $N$ . For simplicity, we denote  $\tau_{2,0}(f)$  by  $\tau_2(f)$ .

By decomposing the left-hand side of (9.2) into its tangential and normal components, we have

**Proposition 9.1** ([4]). *Let  $M$  be an  $n$ -dimensional submanifold of  $\tilde{R}^m(\epsilon)$ . Then  $M$  is  $\lambda$ -biharmonic if and only if*

$$\begin{cases} \Delta^D H + \text{trace } B(\cdot, A_H(\cdot)) + (\lambda - \epsilon n)H = 0, \\ 4\text{trace } A_{D(\cdot)} H(\cdot) + n\text{grad}(|H|^2) = 0, \end{cases}$$

where  $\Delta^D = -\sum_{i=1}^n \{D_{e_i} D_{e_i} - D_{\nabla_{e_i} e_i}\}$ .

**Proposition 9.2** ([25]). *Let  $M$  be an  $n$ -dimensional submanifold of  $\tilde{M}^m(4\epsilon)$  such that  $JH$  is tangent to  $M$ . Then  $M$  is  $\lambda$ -biharmonic if and only if*

$$\begin{cases} \Delta^D H + \text{trace } B(\cdot, A_H(\cdot)) + \{\lambda - \epsilon(n+3)\}H = 0, \\ 4\text{trace } A_{D(\cdot)H}(\cdot) + n\text{grad}(|H|^2) = 0. \end{cases}$$

**Remark 9.1.** By Proposition 9.2, we see that all hypersurfaces with constant principal curvatures in  $R^m(\epsilon)$  and  $\tilde{M}^m(4\epsilon)$  are  $\{-|B|^2 + \epsilon(m-1)\}$ -biharmonic and  $\{-|B|^2 + 2\epsilon(m+1)\}$ -biharmonic, respectively.

It follows from (9.1) and (9.2) that any minimal submanifold is  $\lambda$ -biharmonic. Thus, it is interesting to investigate non-minimal  $\lambda$ -biharmonic submanifolds.

**Remark 9.2.** Let  $f : M \rightarrow \mathbb{R}^n$  be an isometric immersion. We denote the mean curvature vector field of  $M$  by  $H = (H_1, \dots, H_n)$ . Then, it follows from (9.1) and (9.2) that  $M$  is  $\lambda$ -biharmonic if and only if it satisfies

$$\Delta_M H_i = -\lambda H_i, \quad 1 \leq i \leq n, \quad (9.3)$$

where  $\Delta_M$  is the Laplace operator acting on  $C^\infty(M)$ . Hence, the notion of biharmonic submanifolds in Definition 9.2. is same as one defined by B. Y. Chen (cf. [17]). It was proved in [10] that a submanifold  $M$  satisfies (9.3) if and only if one of the following three cases occurs:

- (1)  $f$  satisfies  $\Delta_M f = -\lambda f$ ;
- (2)  $f$  can be written as  $f = f_0 + f_1$ ,  $\Delta_M f_0 = 0$ ,  $\Delta_M f_1 = -\lambda f_1$ ;
- (3)  $M$  is a biharmonic submanifold.

An immersion described in (1) (resp. (2)) is said to be of *1-type* (resp. *null 2-type*). An immersion  $f : M \rightarrow \mathbb{R}^n$  is of 1-type if and only if either  $M$  is a minimal submanifold of  $\mathbb{R}^n$  or  $M$  is a minimal submanifold of a hypersphere in  $\mathbb{R}^n$  (cf. [12, Theorem 3.2]). The classification of null 2-type immersions is not yet complete.

There exist many non-minimal biharmonic submanifolds in a sphere or a complex projective space (see, for example, [5] and [25]). On the other hand, the following conjecture proposed by Chen [12] is still open.

**Conjecture 9.1.** *Any biharmonic submanifold in Euclidean space is minimal.*

Several partial positive answers to this conjecture have been obtained (see [17]). For example, Chen and Munteanu [18] proved that Conjecture 9.1 is true for hypersurfaces which are  $\delta(2)$ -ideal or  $\delta(3)$ -ideal in Euclidean space of arbitrary dimension.

As an extension of the notion of biharmonic submanifolds, the following notion was introduced by Loubeau and Montaldo in [36].

**Definition 9.3.** An isometric immersion  $f : M \rightarrow N$  is called a  $\lambda$ -*biminimal* if it is a critical point of the  $\lambda$ -bienergy functional with respect to all *normal variations* with compact support. Here, a normal variation means a variation  $f_t$  through  $f = f_0$  such that the variational vector field  $V = df_t/dt|_{t=0}$  is normal to  $f(M)$ . In this case,  $M$  is called a  $\lambda$ -*biminimal submanifold* in  $N$ . In the case of  $\lambda = 0$ , we simply call it *biminimal* submanifold.

An isometric immersion  $f$  is  $\lambda$ -biminimal if and only if

$$[\tau_{2,\lambda}(f)]^\perp = 0,$$

where  $[\cdot]^\perp$  denotes the normal component of  $[\cdot]$  (see [36]). It is known that there exist ample examples of  $\lambda$ -biminimal submanifolds in real and complex space forms, which are not  $\lambda$ -biharmonic (see, for example, [36], [43], [45] and [47]).

In [44], the notion of tangentially biharmonicity for submanifolds was introduced as follows:

**Definition 9.4.** Let  $f : M \rightarrow N$  be an isometric immersion. Then  $M$  is called a *tangentially biharmonic submanifold* in  $N$  if it satisfies

$$[\tau_2(f)]^\top = 0, \quad (9.4)$$

where  $[\cdot]^\top$  denotes the tangential part of  $[\cdot]$ .

**Example 9.1.** Let  $x : M^{n-1} \rightarrow \mathbb{R}^n$  be an isometric immersion. The normal bundle  $T^\perp M^{n-1}$  of  $M^{n-1}$  is naturally immersed in  $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$  by the immersion  $f(\xi_x) := (x, \xi_x)$ , which is expressed as

$$f(x, s) = (x, sV) \quad (9.5)$$

for the unit normal vector field  $V$  along  $x$ . We equip  $T^\perp M^{n-1}$  with the metric induced by  $f$ . If we define the complex structure  $J$  on  $\mathbb{C}^n = \mathbb{R}^n \times \mathbb{R}^n$  by  $J(X, Y) := (-Y, X)$ , then  $T^\perp M^{n-1}$  is a Lagrangian submanifold in  $\mathbb{C}^n$  (see [26, III.3.C]). It was proved in [44] that  $T^\perp M^2$  is a tangentially biharmonic Lagrangian submanifold in  $\mathbb{C}^3$  if and only if  $M^2$  is either minimal, a part of a round sphere or a part of a circular cylinder in  $\mathbb{R}^3$ .

**Remark 9.3.** For any  $\lambda \in \mathbb{R}$ , we have  $[\tau_{2,\lambda}(f)]^\top = [\tau_2(f)]^\top$ .

**Remark 9.4.** By the first variation formula for  $E_2$  obtained in [30], we see that an isometric immersion  $f : M \rightarrow N$  is tangentially biharmonic if and only if it is a critical point of  $E_2$  with respect to all *tangential variations* with compact support. Here, a tangential variation means a variation  $f_t$  through  $f = f_0$  such that the variational vector field  $V = df_t/dt|_{t=0}$  is tangent to  $f(M)$ .

**Remark 9.5.** As described by Hilbert [28], the stress-energy tensor associated to a variational problem is a symmetric 2-covariant tensor which is conservative, namely, divergence-free at critical points. The stress-energy tensor  $S_2$  for  $E_2(f)$  was introduced by Jiang [31] as follows:

$$\begin{aligned} S_2(X, Y) = & \frac{1}{2} |\tau(f)|^2 \langle X, Y \rangle + \langle df, \nabla^f \tau(f) \rangle \\ & - \langle df(X), \nabla_Y^f \tau(f) \rangle - \langle df(Y), \nabla_X^f \tau(f) \rangle. \end{aligned}$$

It satisfies  $\operatorname{div} S_2 = -\langle \tau_2(f), df \rangle$ . Hence, an isometric immersion  $f$  is tangentially biharmonic if and only if  $\operatorname{div} S_2 = 0$ . Caddeo et al. [9] called these submanifolds satisfying such a condition as *biconservative submanifolds*, and moreover, classified biconservative surfaces in 3-dimensional real space forms.



**Remark 9.6.** Hasanis and Vlachos [27] classified hypersurfaces in  $\mathbb{R}^4$  satisfying (9.4). They called such hypersurfaces as *H-hypersurfaces*. Afterwards, the biharmonic ones are picked out in the class. As a result, the non-existence of non-minimal biharmonic hypersurfaces in  $\mathbb{R}^4$  was proved.

**Remark 9.7.** It follows from Definition 9.3 and 9.4 that a map  $f$  is  $\lambda$ -biharmonic (resp.  $\lambda$ -biminimal) if and only if it is a critical point of  $E_2(f)$  for all variations (resp. normal variations) with compact support and *fixed energy*. Here,  $\lambda$  is the Lagrange multiplier.

#### 9.4 Biharmonic ideal CR submanifolds

For homogeneous real hypersurfaces in  $\mathbb{C}P^m(4)$ , namely, orbits under some subgroups of the projective unitary group  $PU(m+1)$ , we have

**Theorem 9.1** ([29]). *Let  $M$  be a homogeneous hypersurface in  $\mathbb{C}P^m(4)$ . Then,  $M$  is non-minimal biharmonic if and only if it is congruent to an open portion of one of the following real hypersurfaces:*

- (1) *a tubular hypersurface over  $\mathbb{C}P^q(4)$  with radius*

$$r = \cot^{-1} \left( \sqrt{\frac{m+2 \pm \sqrt{(2q-m+1)^2 + 4(m+1)}}{2m-2q-1}} \right).$$

- (2) *a tubular hypersurface over the Plücker imbedding of the complex Grassmann manifold  $Gr_2(\mathbb{C}^5) \subset \mathbb{C}P^9(4)$  with radius  $r$ , where  $0 < r < \pi/4$  and  $t = \cot r$  is a unique solution of the equation*

$$41t^6 + 43t^4 + 41t^2 - 15 = 0.$$

*In this case,  $r = 1.0917 \dots$ .*

- (3) *a tubular hypersurface over the canonical imbedding of the Hermitian symmetric space  $SO(10)/U(5) \subset \mathbb{C}P^{15}(4)$  with radius  $r$ , where  $0 < r < \pi/4$  and  $t = \cot r$  is a unique solution of the equation*

$$13t^6 - 107t^4 + 43t^2 - 9 = 0.$$

*In this case,  $r = 0.343448 \dots$ .*

For details on the canonical imbedding of a compact Hermitian symmetric space into  $\mathbb{C}P^m(4)$ , we refer the reader to Section 4 of [39].

**Remark 9.8.** Let  $M$  be a real hypersurface in  $\mathbb{C}P^m(4)$ . Kimura [32] proved that  $M$  is a Hopf hypersurface with constant principal curvatures if and only if it is homogeneous.

Combining Theorem 6.1, Proposition 9.2 and Theorem 9.1, we obtain

**Corollary 9.1.** *Let  $M$  be a  $\delta_k(2)$ -ideal non-minimal biharmonic Hopf hypersurface in  $\mathbb{C}P^m(4)$ . Then,  $m$  is odd and  $M$  is an open portion of a tubular hypersurface over  $\mathbb{C}P^{(m-1)/2}(4)$  with radius*

$$r = \cot^{-1} \left( \sqrt{\frac{m+2 \pm 2\sqrt{m+1}}{m}} \right).$$

**Example 9.2.** On each CR submanifold described in Theorem 5.4, there exists an orthonormal frame  $\{e_1, \dots, e_{2m}\}$  such that  $e_{2r} = Je_{2r-1}$  for  $r \in \{1, \dots, n\}$ ,  $JH \parallel e_{2n+1} \in \mathcal{H}^\perp$  and the second fundamental form  $B$  takes the following form:

$$\begin{aligned} B(e_{2r-1}, e_{2r-1}) &= \sqrt{\frac{k}{2n-k}} Je_{2n+1} + \phi_r \xi_r, \\ B(e_{2r}, e_{2r}) &= \sqrt{\frac{k}{2n-k}} Je_{2n+1} - \phi_r \xi_r, \\ B(e_{2r-1}, e_{2r}) &= \phi_r J\xi_r, \\ B(e_{2n+1}, e_{2n+1}) &= \frac{2n}{\sqrt{k(2n-k)}} Je_{2n+1}, \\ B(u_i, u_j) &= h(u_i, e_{2n+1}) = 0 \quad (i \neq j), \end{aligned}$$

where  $\phi_r$  are functions,  $\xi_r \in \nu$  and  $u_j \in L_j$  (see Lemma 7 of [42]). Here,  $\nu$  denotes an orthogonal complement of  $J\mathcal{H}^\perp$  in  $T^\perp M$ . Therefore, by using Proposition 9.2, we find that all ideal CR submanifolds given in Theorem 5.4 are non-minimal  $\lambda$ -biharmonic submanifolds with

$$\lambda = -\frac{2n(2n+k^2)}{k(2n-k)} - 2n - 4 \quad (\neq 0).$$

The following problem seems interesting.

**Problem 9.1.** *Classify  $\delta(n_1, \dots, n_k)$ -ideal proper CR submanifolds in  $\mathbb{C}P^m(4)$  which are non-minimal biharmonic.*

**Example 9.3.** The standard product  $S^{2r+1}(1/\sqrt{2}) \times S^{2s+1}(1/\sqrt{2})$  in  $S^{2(r+s)+3}(1)$  is a biharmonic contact CR hypersurface (see [48, Example 5.1] and [5, p.92]). Its principal curvatures are  $\{1, -1\}$  with multiplicities  $\{2r+1, 2s+1\}$ . We may assume that  $r \geq s$ . By Theorem 4.1, we see that the biharmonic hypersurface  $S^{2r+1}(1/\sqrt{2}) \times S^{2s+1}(1/\sqrt{2})$  is minimal and  $\delta_{2s+1}(2)$ -ideal if  $r = s$ ; otherwise it is non-minimal and  $\delta(4s+3)$ -ideal.

Incidentally, the following conjectures proposed in [4] remains open.

**Conjecture 9.2.** *The only non-minimal biharmonic hypersurfaces in  $S^{m+1}$  are the open parts of hyperspheres  $S^m(1/\sqrt{2})$  or of the standard products  $S^{m_1}(1/\sqrt{2}) \times S^{m_2}(1/\sqrt{2})$ , where  $m_1 + m_2 = m$  and  $m_1 \neq m_2$ .*

**Conjecture 9.3.** *Any non-minimal biharmonic submanifold in  $S^n(1)$  has constant mean curvature.*

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